

# Math 2010 Week 10

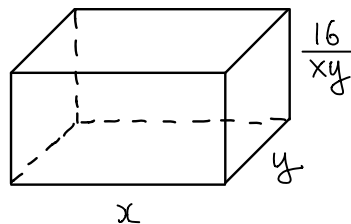
Another example of extrema on unbounded region

eg Make a box (without top) with volume = 16

Cost:

Base \$2/unit area

Side \$0.5/unit area



Q How to minimize cost?

Sol Want to minimize

$$C(x,y) = 2xy + \left(\frac{16}{xy}x + \frac{16}{xy}y\right)(2)(0.5)$$
$$= 2xy + \frac{16}{x} + \frac{16}{y}$$

on the domain  $\Omega = \{(x,y) \in \mathbb{R}^2 : x,y > 0\}$

- $\Omega$  is neither closed nor bounded.  
∴ EVT cannot be applied directly
- $C$  is large if  $x$  or  $y$  is small or large.

Strategy: Find a rectangle  $R$  s.t.  
 $C > \min. \text{ of } C|_R$  on  $\partial R$  and outside  $R$ .

Step 1 Find critical points

$$\nabla C = \left(2y - \frac{16}{x^2}, 2x - \frac{16}{y^2}\right) \text{ exists everywhere}$$

$$\nabla C = \vec{0} \iff \begin{cases} 2y - \frac{16}{x^2} = 0 \\ 2x - \frac{16}{y^2} = 0 \end{cases}$$

$$\therefore y = \frac{8}{x^2}, x = \frac{8}{y^2} = \frac{8}{\frac{64}{x^4}} = \frac{x^4}{8}, x > 0$$

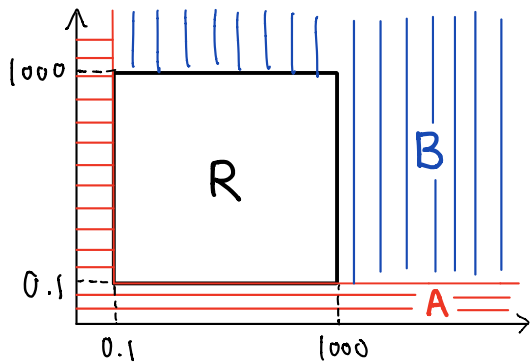
$$\Rightarrow x^3 = 8, x = 2, y = 2$$

∴ Only one critical point  $(2,2)$ ,  $C(2,2) = 24$

Step 2 Choose  $R$  s.t.  $C > 24$  on  $\partial R$  and outside  $R$ .

$$C(x,y) = 2xy + \frac{16}{x} + \frac{16}{y}$$

One possible choice:  $R = [0.1, 1000] \times [0.1, 1000]$



**(A)** If  $x \leq 0.1$  or  $y \leq 0.1$

$$\text{then } C > \frac{16}{x} + \frac{16}{y} > \frac{16}{0.1} = 160 > 24$$

**(B)** If  $(x \geq 0.1, y \geq 1000)$  or  $(y \geq 0.1, x \geq 1000)$ ,

$$\text{then } C > 2(0.1)(1000) = 200 > 24$$

Step 3 Analysis

•  $R$  is closed and bounded,  $C$  is continuous

By EVT,  $C|_R$  has minimum

•  $C$  has only one critical point  $(2,2) \in \Omega$

$$(2,2) \in \text{int}(R), C(2,2) = 24$$

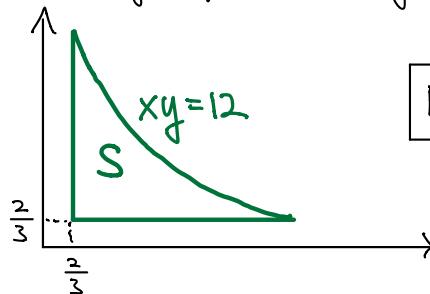
$C > 24$  on  $\partial R$

$\Rightarrow C|_R$  has min value 24 at  $(2,2)$

•  $C > 24$  outside  $R$

$\Rightarrow C$  has min value 24 at  $(2,2)$  on  $\Omega$

Rmk One may replace  $R$  by  $S$  below.



Exercise

## Taylor Series Expansion (Thomas 14.9)

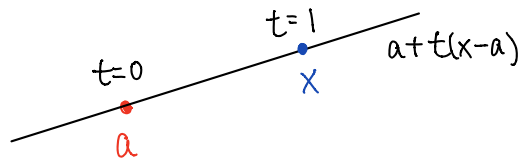
Recall Taylor expansion for 1-variable function

$g(t)$  at  $t=0$  up to order  $k$

$$g(t) = g(0) + g'(0)t + \frac{1}{2!} g''(0)t^2 + \dots \\ + \frac{1}{k!} g^{(k)}(0)t^k + \text{remainder} \quad (*)$$

We want a similar formula for a multi-variable function  $f(x)$  defined near  $a$ , where  $x = (x_1, \dots, x_n)$   $a = (a_1, \dots, a_n)$

Let  $g(t) = f(a + t(x-a))$



If  $\|x-a\|$  is small, then for  $|t| \leq 1$ ,

$\|t(x-a)\| = |t|\|x-a\| \leq \|x-a\|$  is small

and  $g(t)$  is defined

By  $(*)$ ,

$$f(a + t(x-a)) = g(0) + g'(0)t + \frac{1}{2!} g''(0)t^2 + \dots \\ + \frac{1}{k!} g^{(k)}(0)t^k + \text{remainder}$$

Put  $t=1$ ,

$$f(x) = g(0) + g'(0) + \frac{1}{2!} g''(0) + \dots + \frac{1}{k!} g^{(k)}(0) \\ + \text{remainder}$$

Next, express  $g^{(k)}(0)$  in terms of  $f$ :

$$g(0) = f(a + t(x-a)) = f(a)$$

$$g'(t) = \nabla f(a + t(x-a)) \cdot \frac{d}{dt} (a + t(x-a))$$

$$= \nabla f(a + t(x-a)) \cdot (x-a)$$

$$= \sum_{i=1}^n \frac{\partial f}{\partial x_i} (a + t(x-a)) (x_i - a_i)$$

$$\Rightarrow g'(0) = \nabla f(a) \cdot (x-a)$$

$$= \sum_{i=1}^n \frac{\partial f}{\partial x_i} (a) (x_i - a_i)$$

$$g''(t) = \frac{d}{dt} g'(t)$$

$$= \sum_{i=1}^n \frac{d}{dt} \left[ \frac{\partial f}{\partial x_i} (a + t(x-a)) (x_i - a_i) \right]$$

$$= \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x_j \partial x_i} (a + t(x-a)) (x_j - a_j) (x_i - a_i)$$

$$g''(0) = \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x_j \partial x_i} (a) (x_j - a_j) (x_i - a_i)$$

$\therefore$  Taylor Expansion at  $a$  up to order 2 is

$$f(x) = f(a) + \sum_{i=1}^n \frac{\partial f}{\partial x_i} (a) (x_i - a_i) + \frac{1}{2!} \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j} (a) (x_i - a_i) (x_j - a_j) + \text{remainder}$$

eg If  $n=2$ , i.e.  $f = f(x, y)$ ,  $a = (x_0, y_0)$

$f$  is  $C^2$  (so  $f_{xy} = f_{yx}$ ), then

$$f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + \frac{1}{2} \left[ f_{xx}(x_0, y_0)(x - x_0)^2 + 2f_{xy}(x_0, y_0)(x - x_0)(y - y_0) + f_{yy}(x_0, y_0)(y - y_0)^2 \right]$$

+ remainder.

Similarly, the general term is

$$g^{(k)}(0) = \sum_{i_1, \dots, i_k=1}^n \frac{\partial^k f}{\partial x_{i_1} \dots \partial x_{i_k}} (a) (x_{i_1} - a_{i_1}) \dots (x_{i_k} - a_{i_k})$$



## Thm (Taylor Theorem)

Let  $\Omega \subseteq \mathbb{R}^n$  be open,  $f: \Omega \rightarrow \mathbb{R}$  be  $C^k$ .

Then for any  $x, a \in \Omega$ ,

$$f(x) = f(a) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(a)(x_i - a_i) + \frac{1}{2!} \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(a)(x_i - a_i)(x_j - a_j) + \dots$$
$$+ \frac{1}{k!} \sum_{i_1, \dots, i_k=1}^n \frac{\partial^k f}{\partial x_{i_1} \dots \partial x_{i_k}}(a)(x_{i_1} - a_{i_1}) \dots (x_{i_k} - a_{i_k}) + \varepsilon_k(x, a)$$

with  $\lim_{x \rightarrow a} \frac{\varepsilon_k(x, a)}{\|x - a\|^k} = 0$

## Defn

$$p_k(x) = f(a) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(a)(x_i - a_i) + \dots$$
$$+ \frac{1}{k!} \sum_{i_1, \dots, i_k=1}^n \frac{\partial^k f}{\partial x_{i_1} \dots \partial x_{i_k}}(a)(x_{i_1} - a_{i_1}) \dots (x_{i_k} - a_{i_k})$$

is called the  $k$ -th order Taylor polynomial of  $f$  at  $a$

## Rmk

①  $p_1(x) = L(x)$

= Linearization of  $f$  at  $a$

②  $p_k$  and  $f$  have equal partial derivatives up to order  $k$  at  $a$

eg  $f(x,y) = e^x \cos y$

Find 2<sup>nd</sup> order Taylor polynomial at  $a = (0,0)$

Sol  $f_x = e^x \cos y$        $f_y = -e^x \sin y$

$$f_{xx} = e^x \cos y$$
$$f_{yx} = -e^x \sin y$$

$$f_{xy} = -e^x \sin y$$
$$f_{yy} = -e^x \cos y$$

$$\Rightarrow f(0,0) = 1,$$

$$f_x(0,0) = 1, \quad f_y(0,0) = 0$$

$$f_{xx}(0,0) = 1 \quad f_{yy}(0,0) = -1$$

$$f_{xy}(0,0) = f_{yx}(0,0) = 0$$

$$p_2(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y$$

$$+ \frac{1}{2!} (f_{xx}(0,0)x^2 + 2f_{xy}(0,0)xy + f_{yy}(0,0)y^2)$$

$$= 1 + x + \frac{1}{2}x^2 - \frac{1}{2}y^2$$

How about  $p_3(x,y)$  at  $(0,0)$ ?

$$p_3(x,y) = p_2(x,y) + \underbrace{\frac{1}{3!} g^{(3)}(0)}_{3^{\text{rd}} \text{ order terms}}$$

$$f_{xxx} = e^x \cos y$$

$$f_{xxy} = f_{xyx} = f_{yxx} = -e^x \sin y$$

$$f_{xyy} = f_{yxy} = f_{yyx} = -e^x \cos y$$

$$f_{yyy} = e^x \sin y$$

$$\Rightarrow f_{xxx}(0,0) = 1 \quad f_{xxy}(0,0) = 0$$

$$f_{xyy}(0,0) = -1 \quad f_{yyy}(0,0) = 0$$

$$g^{(3)}(0) = f_{xxx}(0,0)x^3 + 3f_{xxy}(0,0)x^2y$$
$$+ 3f_{xyy}(0,0)xy^2 + f_{yyy}(0,0)y^3$$

$$= x^3 - 3xy^2$$

$$p_3(x,y) = p_2(x,y) + \frac{1}{3!} (x^3 - 3xy^2)$$

$$= 1 + x + \frac{1}{2}x^2 - \frac{1}{2}y^2 + \frac{1}{6}x^3 - \frac{1}{2}xy^2$$

Q If  $f = f(x, y, z)$  is  $C^6$ , then

Coefficient of  $xy^2z^3$  in  $p_6(x, y, z)$  at  $(0, 0, 0)$

is  $\alpha f_{xyyzzz}(0, 0, 0)$ .  $\alpha = ?$

Rmk A General Taylor's formula for  $f(x, y)$  is given on P822 of Thomas' Calculus.

Matrix form for 2<sup>nd</sup> order Taylor Polynomial

Defn Let  $\Omega \subseteq \mathbb{R}^n$  be open,  $f: \Omega \rightarrow \mathbb{R}$  be  $C^2$ .

Then the Hessian matrix of  $f$  at  $a \in \Omega$  is

$$Hf(a) = \begin{bmatrix} f_{x_1x_1}(a) & \cdots & f_{x_1x_n}(a) \\ \vdots & \ddots & \vdots \\ f_{x_nx_1}(a) & \cdots & f_{x_nx_n}(a) \end{bmatrix}$$

Rmk

①  $Hf(a)$  is a symmetric  $n \times n$  matrix by the mixed derivatives theorem.

② In Thomas' Calculus, Hessian of  $f$  is defined to be the determinant of our Hessian matrix

With the Hessian matrix, the 2nd order Taylor polynomial of  $f$  at  $a$  can be written as

$$p_2(x) = f(a) + \nabla f(a)(x-a) + \frac{1}{2}(x-a)^T Hf(a)(x-a)$$

1x1    1x1    1xn    nx1    1xn    nxn    nx1

where  $x, a \in \mathbb{R}^n$  are written as column vectors

$$\begin{aligned} (x-a)^T &= \text{Transpose of } x-a \\ &= [x_1 - a_1, \dots, x_n - a_n] \end{aligned}$$

Rmk

$$(x-a)^T Hf(a)(x-a) = [x_1-a_1, \dots, x_n-a_n] \begin{bmatrix} f_{x_1 x_1}(a) & \dots & f_{x_1 x_n}(a) \\ \vdots & \ddots & \vdots \\ f_{x_n x_1}(a) & \dots & f_{x_n x_n}(a) \end{bmatrix} \begin{bmatrix} x_1-a_1 \\ \vdots \\ x_n-a_n \end{bmatrix}$$

$$= [x_1-a_1, \dots, x_n-a_n] \begin{bmatrix} f_{x_1 x_1}(a)(x_1-a_1) + \dots + f_{x_1 x_n}(a)(x_n-a_n) \\ \vdots \\ f_{x_n x_1}(a)(x_1-a_1) + \dots + f_{x_n x_n}(a)(x_n-a_n) \end{bmatrix}$$

$$= f_{x_1 x_1}(a)(x_1-a_1)(x_1-a_1) + \dots + f_{x_1 x_n}(a)(x_1-a_1)(x_n-a_n)$$

+ ...

⋮

$$+ f_{x_n x_1}(a)(x_1-a_1)(x_n-a_n) + \dots + f_{x_n x_n}(a)(x_n-a_n)(x_n-a_n)$$

$$= \sum_{i,j=1}^n f_{x_i x_j}(a)(x_i-a_i)(x_j-a_j)$$

$$= g^{(2)}(0)$$

eg Same  $f(x,y) = e^x \cos y$ .

Find  $p_2(x,y)$  at  $a=(0,0)$

using matrix form.

Sol  $f(0,0) = 1$

$$\nabla f(0,0) = (1, 0)$$

$$Hf(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$p_2(x,y) = f(0,0) + \nabla f(0,0) \begin{bmatrix} x-0 \\ y-0 \end{bmatrix}$$

$$+ \frac{1}{2} [x-0 \ y-0] Hf(0,0) \begin{bmatrix} x-0 \\ y-0 \end{bmatrix}$$

$$= 1 + [1 \ 0] \begin{bmatrix} x \\ y \end{bmatrix} + \frac{1}{2} [x \ y] \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= 1 + x + \frac{1}{2}x^2 - \frac{1}{2}y^2$$

eg  $g(x,y) = \frac{\ln x}{1-y}$ . Find  $p_2(x,y)$  at  $(1,0)$

Sol  $g(1,0) = 0$

$$\nabla g = [g_x, g_y] = \left[ \frac{1}{x(1-y)}, \frac{\ln x}{(1-y)^2} \right]$$

$$Hg = \begin{bmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{bmatrix} = \begin{bmatrix} -\frac{1}{x^2(1-y)} & \frac{1}{x(1-y)^2} \\ \frac{1}{x(1-y)^2} & \frac{2 \ln x}{(1-y)^3} \end{bmatrix}$$

$$\nabla g(1,0) = [1 \ 0] \quad Hg(1,0) = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$p_2(x,y)$$

$$= g(1,0) + \nabla g(1,0) \begin{bmatrix} x-1 \\ y \end{bmatrix} + \frac{1}{2} [x-1 \ y] Hg(1,0) \begin{bmatrix} x-1 \\ y \end{bmatrix}$$

$$= 0 + [1 \ 0] \begin{bmatrix} x-1 \\ y \end{bmatrix} + \frac{1}{2} [x-1 \ y] \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x-1 \\ y \end{bmatrix}$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + (x-1)y$$

## Application to local max/min.

If  $f$  is  $C^2$ ,  $a$  is a critical point of  $f$

Then  $\nabla f(a) = \vec{0}$ . For  $x$  near  $a$ ,

$$f(x) \approx p_2(x)$$

$$= f(a) + \nabla f(a)(x-a) + \frac{1}{2}(x-a)^T Hf(a)(x-a)$$

$$= f(a) + \underbrace{\frac{1}{2}(x-a)^T Hf(a)(x-a)}$$

This term determines whether  
 $f(x) > f(a)$  or  $f(x) < f(a)$

Rmk For  $n=1$ , i.e.  $f$  is 1-variable.

$$\frac{1}{2}(x-a)^T Hf(a)(x-a) = \frac{1}{2} f''(a)(x-a)^2$$

If  $f'(a) = 0$ , then

$$\begin{cases} f''(a) > 0 \Rightarrow \text{local min at } a & \left( \begin{array}{l} 2^{\text{nd}} \text{ derivative} \\ \text{test} \end{array} \right) \\ f''(a) < 0 \Rightarrow \text{local max at } a \end{cases}$$

For  $n=2$ , the 2<sup>nd</sup> order term is

$$\frac{1}{2} \begin{bmatrix} x-x_0 & y-y_0 \end{bmatrix} \underbrace{\begin{bmatrix} f_{xx}(x_0, y_0) & f_{xy}(x_0, y_0) \\ f_{yx}(x_0, y_0) & f_{yy}(x_0, y_0) \end{bmatrix}}_{\text{Symmetric}} \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix}$$

$f$  is  $C^2 \Rightarrow$  Symmetric

To understand nature of critical points, we study quadratic forms of 2 variables.

$$\begin{aligned} q(x, y) &= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= Ax^2 + 2Bxy + Cy^2 \end{aligned}$$

Does  $q(x, y)$  have a definite sign (always positive or always negative) for  $(x, y) \neq (0, 0)$ ?

We can determine it by completing square.

eg 1  $q(x,y) = 2xy$

Note  $q(x,y) = \frac{1}{2}(x+y)^2 - \frac{1}{2}(x-y)^2$

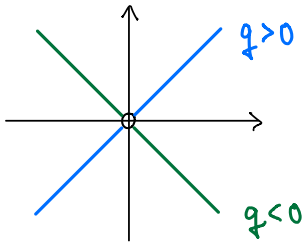
Along  $x+y=0$ , i.e.  $y=-x$ ,

$$q(x,-x) = -2x^2 < 0 \quad \text{for } x \neq 0$$

Along  $x-y=0$ , i.e.  $y=x$

$$q(x,x) = 2x^2 > 0 \quad \text{for } x \neq 0$$

$\therefore q$  has no definite sign, i.e. indefinite



Clearly  $(0,0)$  is a critical point of  $q(x,y)$  but neither local max nor min. Such a critical point is called a saddle point.

eg 2  $q(x,y) = 17x^2 - 12xy + 8y^2$ . Definite sign?

Soln

$$q(x,y) = 17 \left[ x^2 - \frac{2 \cdot 6}{17} xy + \left(\frac{6}{17}\right)^2 y^2 \right] + \left(8 - \frac{36}{17}\right) y^2$$

$$= 17 \left( x - \frac{6}{17} y \right)^2 + \frac{100}{17} y^2 \quad \otimes$$

$\therefore q(x,y) > 0 = q(0,0)$  for  $(x,y) \neq (0,0)$

$\therefore$  The critical point  $(0,0)$  is a local min.  
also global min of  $q(x,y)$

Rmk Expression like  $\otimes$  is called diagonalization of quadratic form. It is not unique! For

example  $q(x,y) = 5 \left( \frac{x+2y}{\sqrt{5}} \right)^2 + 20 \left( \frac{2x-y}{\sqrt{5}} \right)^2$

is another diagonalization.  $\nwarrow \nearrow$   
"Orthogonal" change of coordinates

## Higher dimension example

eg 3  $q(x, y, z) = xy + yz + zx$

Definite sign for  $(x, y, z) \neq (0, 0, 0)$ ?

Sol  $q = \frac{1}{4}(x+y)^2 - \frac{1}{4}(x-y)^2 + z(x+y)$

Let  $u = \frac{x+y}{2}$   $v = \frac{x-y}{2}$ . Then

$$\begin{aligned} q &= u^2 - v^2 + 2uz \\ &= (u^2 + 2uz + z^2) - v^2 - z^2 \\ &= (u+z)^2 - v^2 - z^2 \\ &= \left(\frac{x+y}{2} + z\right)^2 - \left(\frac{x-y}{2}\right)^2 - z^2 \\ &= \frac{1}{4}(x+y+2z)^2 - \frac{1}{4}(x-y)^2 - z^2 \end{aligned}$$

↑ positive ↖ ↗ negative

On the plane  $x+y+2z=0$ , i.e.  $z = -\frac{x+y}{2}$

$$q = q\left(x, y, -\frac{x+y}{2}\right)$$

$$= -\frac{1}{4}(x-y)^2 - \frac{1}{4}(x+y)^2 < 0 \text{ for } (x, y, z) \neq (0, 0, 0)$$

Along the line  $x-y=z=0$ , i.e.  $y=x, z=0$

$$q(x, y, z) = q(x, x, 0)$$

$$= x^2 > 0 \text{ for } x \neq 0$$

∴ The critical point  $(0, 0, 0)$  is a saddle point.

For general theory, need linear algebra:

Diagonalization of quadratic form, eigenvalues...



Defn Let  $A$  be a  $n \times n$  symmetric matrix.

Then  $A$  is said to be

- ① positive definite if  $x^T A x > 0$   
for all column vectors  $x \in \mathbb{R}^n \setminus \{\vec{0}\}$
- ② negative definite if  $x^T A x < 0$   
for all column vectors  $x \in \mathbb{R}^n \setminus \{\vec{0}\}$
- ③ indefinite if  $\exists$  column vectors  $x, y \in \mathbb{R}^n \setminus \{\vec{0}\}$   
such that  $x^T A x > 0$  and  $y^T A y < 0$

Rmk These are not all the possible cases:

There are symmetric matrix which is not positive definite, negative definite nor indefinite.

eg

$$\textcircled{1} \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x^2 + 4y^2 > 0 \quad \forall \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \setminus \{\vec{0}\}$$

$\therefore \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$  is positive definite

$$\textcircled{2} \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -x^2 - 4y^2 < 0 \quad \forall \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \setminus \{\vec{0}\}$$

$\therefore \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}$  is negative definite

$$\textcircled{3} \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -x^2 + 4y^2$$

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -1 < 0$$

$$\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 4 > 0$$

$\therefore \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$  is indefinite

$$\textcircled{4} \quad [x \ y] \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x^2 \geq 0 \quad \forall \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$$

$$[0 \ 1] \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \Rightarrow \text{not positive definite}$$

$\therefore \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$  is neither positive/negative definite  
nor indefinite

$$\textcircled{5} \quad [x \ y] \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= x^2 + 4xy + 5y^2$$

$$= (x^2 + 4xy + 4y^2) + y^2$$

$$= (x + 2y)^2 + y^2 > 0 \quad \forall \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \setminus \{\vec{0}\}$$

$\therefore \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$  is positive definite.